Integrable three-dimensional systems possessing linear and quadratic integrals of motion in the momenta

1987 J. Phys. A: Math. Gen. 20 5079

(http://iopscience.iop.org/0305-4470/20/15/024)

J. Phys. A: Math. Gen. 20 (1987) 5079-5087. Printed in the UK

Integrable three-dimensional systems possessing linear and quadratic integrals of motion in the momenta

S Ichtiaroglou

Department of Theoretical Mechanics, University of Thessaloniki, Greece

Received 23 February 1987, in final form 5 May 1987

Abstract. The total number of three-dimensional integrable potentials which possess one linear and one quadratic integral of motion in involution in the momenta is found. These potentials, which are given by a Darboux equation, belong to four distinct classes and involve two arbitrary functions in their expression.

1. Introduction

As is well known, an N-dimensional Hamiltonian system is integrable if N integrals in involution for this system exist (e.g. Arnold 1978). Direct construction of such integrable systems can be achieved by application of a method due to Bertrand (1852). For the two-dimensional case, Darboux (1901) obtained a second-order partial differential equation and found the general form for the potential, in order for the system to possess an integral quadratic in the momenta. Darboux's work is also presented by Whittaker (1937). Very recently, Dorizzi *et al* (1983) completed Darboux's results for some exceptional cases, so that the total number of two-dimensional integrable potentials which possess an integral quadratic in the momenta is known.

Many new two-dimensional integrable systems possessing integrals of more complicated form have also been found by several authors, and all the results in this field up to 1986 can be found in a complete review by Hietarinta (1986).

A different method to check integrability makes use of the Painlevé property of the differential equations of the pertinent system (Bountis *et al* 1982) while non-integrability can be proved in some cases by use of Ziglin's theorem (Ziglin 1983a, b, Yoshida 1986).

Though much work has been done in direct search for two-dimensional integrable systems, there are only a few results in three dimensions. Chandrasekhar (1960) found the general form for the coefficients of an integral of motion quadratic in the velocities possessed by a three-dimensional time-dependent potential and also the set of differential equations which the potential must satisfy. Makarov *et al* (1967) considered the existence of pairs of commuting integrals of motion related to the separation of variables in the Schrödinger equation. Recently, Grammaticos *et al* (1985) presented a method of generalising integrable Hamiltonians from two to N dimensions which is applicable if additional terms can be added to the potential without destroying integrability. Fordy *et al* (1986) studied the integrability of quartic potentials from the point of view of Lax representations. Dorizzi *et al* (1986) found new integrable three-dimensional quartic potentials by combined use of singularity analysis (Painlevé property or Ziglin's theorem) and explicit construction of the integrals of motion.

In this paper, we construct three-dimensional integrable systems which possess one integral which is linear in the momenta. As a first step, we obtain the general form for the potential in order to possess such an integral. Then we find the totality of three-dimensional integrable potentials which possess two integrals linear in the momenta. This result is poor, as expected, since by a suitable transformation these systems simply reduce to two-dimensional central potentials. In § 3 we obtain the total number of three-dimensional integrable potentials which possess one linear and one quadratic integral of motion in the momenta. These systems eventually relate to Darboux's systems and are expressed in all cases via two arbitrary functions.

2. Three-dimensional potentials possessing integrals linear in the momenta

The Hamiltonian for a particle of unit mass moving in space under the influence of a potential V is

$$H=\tfrac{1}{2}\delta_{ij}p_ip_j+V(x_1,x_2,x_3) \qquad (i,j=1,2,3) \tag{1}$$

where x_i are orthogonal cartesian coordinates, $p_i=\dot{x}_i$ and δ_{ij} is the Kronecker delta. (A dot denotes a time derivative while a sum over repeated indices is understood.)

Suppose that system (1) possesses an integral of motion linear in the momenta:

$$I_1=a_i(x_k)p_i=\boldsymbol{a}\cdot\boldsymbol{p}=\text{constant} \qquad (i,k=1,2,3) \tag{2}$$

where $\boldsymbol{a}=a_i\boldsymbol{e}_i$, $\boldsymbol{p}=p_i\boldsymbol{e}_i$ and $\boldsymbol{e}_i$ is the unit vector along the x_i axis. In order that (2) is an integral of motion of system (1), its Poisson bracket with H must vanish identically, i.e.

$$[I_1,H]=\frac{\partial I_1}{\partial x_k}\frac{\partial H}{\partial p_k}-\frac{\partial I_1}{\partial p_k}\frac{\partial H}{\partial x_k}\equiv 0. \tag{3}$$

Equation (3) yields

$$\partial a_i/\partial x_j=-\partial a_j/\partial x_i \qquad (i,j=1,2,3) \tag{4}$$

and also

$$\boldsymbol{a}\cdot\boldsymbol{\nabla}V=0. \tag{5}$$

The solution of system (4) is

$$\boldsymbol{a}=S\boldsymbol{r}+\boldsymbol{d} \tag{6}$$

where $\boldsymbol{r}=x_i\boldsymbol{e}_i$, $\boldsymbol{d}$ is a constant vector and S is a 3×3 skew-symmetric constant matrix. By a suitable translation of the origin of the coordinate system it can be put in the generic case where $S\neq 0$:

$$\boldsymbol{d}=0. \tag{7}$$

In view of equations (6) and (7), I_1 takes the form

$$I_1=(S\boldsymbol{r})\cdot\boldsymbol{p}=(\boldsymbol{s}\times\boldsymbol{r})\cdot\boldsymbol{p} \tag{8}$$

where $\boldsymbol{s}$ is the axial vector of S, i.e. $s_i=\tfrac{1}{2}e_{ijk}S_{jk}$, where e_{ijk} is the permutation symbol. Equation (5) now becomes

$$(\boldsymbol{s}\times\boldsymbol{r})\cdot\boldsymbol{\nabla}V=0 \tag{9}$$

and the solution of (9) for the potential V is

$$V = V(r^2, \boldsymbol{s} \cdot \boldsymbol{r}). \tag{10}$$

Now we can state the following.

Proposition 1. The three-dimensional potentials which possess one integral of motion linear in the momenta are of the form $V = V(r^2, \boldsymbol{s} \cdot \boldsymbol{r})$ where $\boldsymbol{s}$ is a constant vector and the corresponding integral is $I_1 = (\boldsymbol{s} \times \boldsymbol{r}) \cdot \boldsymbol{p}$.

Let us suppose now that system (1) possesses, in addition to I_1, a second integral I_2, linear in the momenta, and also that I_1 and I_2 are in involution, i.e.

$$[I_1, I_2] \equiv 0 \tag{11}$$

so that the system (1) is integrable.

In order that the relation $[I_2, H] \equiv 0$ holds, I_2 must be of the form

$$I_2 = (S'\boldsymbol{r}) \cdot \boldsymbol{p} + \boldsymbol{d}' \cdot \boldsymbol{p} \tag{12}$$

where S' is a 3×3 skew-symmetric constant matrix and $\boldsymbol{d}'$ is a constant vector. If we consider (8) and (12), equation (11) now yields

$$SS' = S'S \tag{13a}$$

and

$$S\boldsymbol{d}' = 0. \tag{13b}$$

Equation (13a) implies

$$S' = k_1 S \tag{14}$$

where k_1 is an arbitrary constant. Indeed, let $\boldsymbol{s}'$ be the axial vector of S'. Since $S'_{ij} = e_{ijk} s'_k$ and $S_{ij} = e_{ijk} s_k$, equation (13a) becomes

$$e_{ikl} e_{kjm} s_l s'_m = e_{ikl} e_{kjm} s'_l s_m$$

or

$$(\delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj}) s_l s'_m = (\delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj}) s'_l s_m$$

or

$$s'_i s_j = s_i s'_j$$

i.e. the vectors $\boldsymbol{s}$ and $\boldsymbol{s}'$ are collinear, which implies equation (14). On the other hand, equation (13b) is written $\boldsymbol{s} \times \boldsymbol{d}' = 0$, i.e.

$$\boldsymbol{d}' = k_2 \boldsymbol{s} \tag{15}$$

where k_2 is also an arbitrary constant. In view of equations (14) and (15), after subtracting $k_1 I_1$ and dividing by k_2, I_2 takes the form

$$I_2 = \boldsymbol{s} \cdot \boldsymbol{p}. \tag{16}$$

Now the condition $[I_2, H] \equiv 0$ is satisfied if also

$$\boldsymbol{s} \cdot \boldsymbol{\nabla} V = 0. \tag{17}$$

Since V is of the form (10), we introduce new variables ξ, η by the relations

$$\xi = r^2 \qquad \eta = \boldsymbol{s} \cdot \boldsymbol{r}$$

and equation (17) takes the form (subscripts denote partial derivatives)

$$2\eta V_\xi + s^2 V_\eta = 0$$

with the solution

$$V = V(s^2\xi - \eta^2). \tag{18}$$

Since

$$s^2\xi - \eta^2 = s^2r^2 - (\boldsymbol{s}\cdot\boldsymbol{r})^2 = |\boldsymbol{s}\times\boldsymbol{r}|^2$$

equation (18) becomes

$$V = V(|\boldsymbol{s}\times\boldsymbol{r}|) \tag{19}$$

so we have proved the following.

Proposition 2. The integrable three-dimensional potentials with two integrals in involution, linear in the momenta, are of the form $V = V(|\boldsymbol{s}\times\boldsymbol{r}|)$ where $\boldsymbol{s}$ is a constant vector. The corresponding integrals are $I_1 = (\boldsymbol{s}\times\boldsymbol{r})\cdot\boldsymbol{p}$ and $I_2 = \boldsymbol{s}\cdot\boldsymbol{p}$.

In order to simplify expression (19), we perform a rotation of the coordinate system so that

$$\boldsymbol{s} = \boldsymbol{e}_3. \tag{20}$$

Then equation (19) takes the form

$$V = V(\rho)$$

where

$$\rho = (x_1^2 + x_2^2)^{1/2} \tag{21}$$

and equations (8) and (16) respectively give

$$I_1 = x_1p_2 - x_2p_1 = l_3 \tag{22a}$$

$$I_2 = p_3 \tag{22b}$$

where l_3 is the component of the angular momentum vector along the x_3 axis. Finally, by a suitable choice of the inertial frame, p_3 can be put equal to zero, and so the systems involved in proposition 2 are in fact two dimensional and correspond to central potentials.

3. Three-dimensional integrable systems possessing one linear and one quadratic integral in the momenta

Let us suppose now that system (1) possesses one integral of motion I_1 linear in the momenta, which, according to § 2, must be of the form (8) and, in addition, one integral I_2, quadratic in the momenta, of the form

$$I_2 = \boldsymbol{p}\cdot(A\boldsymbol{p}) + 2F(x_k) \tag{23}$$

where A is a symmetric matrix, the elements of which depend on x_1, x_2, x_3 as well as the function F.

The condition $[I_2, H] \equiv 0$ leads to the following equations:

$$A_{ij,k} + A_{ik,j} + A_{kj,i} = 0 \tag{24a}$$

and

$$\nabla F = A \nabla V \tag{24b}$$

where

$$A_{ij,k} = (\partial/\partial x_k) A_{ij}.$$

Integrability conditions for (24*b*) are given by the equation

$$\nabla \times (A \nabla V) = 0. \tag{25}$$

The involution condition between the two integrals, $[I_1, I_2] \equiv 0$, implies the equations

$$A_{ij,k} S_{kl} x_l + A_{ik} S_{kj} + A_{jk} S_{ki} = 0 \tag{26a}$$

where S_{ij} are the elements of the constant matrix S appearing in (8), and

$$(\boldsymbol{s} \times \boldsymbol{r}) \cdot \nabla F = 0. \tag{26b}$$

Equation (26*b*) for F coincides with equation (9) for V.

In order to simplify the mathematical procedure, at this point we perform a rotation of the coordinate system so that $\boldsymbol{s} = \boldsymbol{e}_3$. Then we also have $S_{21} = -S_{12} = 1$ while all other S_{ij} equal zero, and I_1 becomes

$$I_1 = x_1 p_2 - x_2 p_1. \tag{27}$$

The solution of equation (9) is of the form given by (10), which can be written for convenience as

$$V = V(\rho, z)$$

where ρ is given by (21) and $z = x_3$.

Function F, as a solution of equation (26*b*), is of the same form

$$F = F(\rho, z).$$

The system of equations (24*a*) has been solved by Chandrasekhar and its solution is (Chandrasekhar 1960, p 96)

$$\begin{aligned}
A_{11} &= -2(h_{20} + h_{21}x_3)x_2 - h_{40}x_2^2 - (a_0 + 2g_{30}x_3 + g_{40}x_3^2) \\
A_{22} &= -2(f_{20} + f_{21}x_1)x_3 - f_{40}x_3^2 - (b_0 + 2h_{30}x_1 + h_{40}x_1^2) \\
A_{33} &= -2(g_{20} + g_{21}x_2)x_1 - g_{40}x_1^2 - (c_0 + 2f_{30}x_2 + f_{40}x_2^2) \\
A_{12} &= A_{21} = (h_{10} + h_{11}x_3 - g_{21}x_3^2) + (h_{20} + h_{21}x_3)x_1 + (h_{30} + f_{21}x_3)x_2 + h_{40}x_1x_2 \\
A_{13} &= A_{31} = (g_{10} + g_{11}x_2 - f_{21}x_2^2) + (g_{20} + g_{21}x_2)x_3 + (g_{30} + h_{21}x_2)x_1 + g_{40}x_1x_3 \\
A_{23} &= A_{32} = (f_{10} + f_{11}x_1 - h_{21}x_1^2) + (f_{20} + f_{21}x_1)x_2 + (f_{30} + g_{21}x_1)x_3 + f_{40}x_2x_3
\end{aligned} \tag{28}$$

where f_{kl}, g_{kl}, h_{kl} $(k = 1, 4,\ l = 0, 1)$, a_0, b_0, c_0 are constants and $f_{11} + g_{11} + h_{11} = 0$. By replacing A_{ij} as given by (28) in equations (26*a*), we obtain the relations

$$\begin{aligned}
&A_{12} = x_1 A_{12,1} = x_2 A_{12,2} \\
&x_1 A_{23,3} = x_2 A_{13,3} \\
&x_2 A_{12,1} - x_1 A_{12,2} + A_{11} - A_{22} = 0 \\
&x_2 A_{13,1} - x_1 A_{13,2} - A_{23} = 0 \\
&x_2 A_{23,1} - x_1 A_{23,2} + A_{13} = 0
\end{aligned}$$

which give

$$f_{21}=g_{21}=h_{21}=0$$

$$f_{10}=g_{10}=h_{10}=0$$

$$h_{11}=f_{30}=h_{30}=g_{20}=h_{20}=0$$

$$f_{40}=g_{40} \qquad f_{11}=-g_{11}$$

$$f_{20}=g_{30} \qquad a_0=b_0.$$

We rename the remaining non-zero constants as follows:

$$c_1=f_{40} \qquad c_2=h_{40} \qquad c_3=f_{11}$$

$$c_4=f_{20} \qquad c_5=a_0 \qquad c_6=c_0.$$

Since V and F are functions of ρ and z only, equations (24b) take the form

$$x_1F_\rho=(x_1A_{11}+x_2A_{12})V_\rho+\rho A_{13}V_z \tag{29a}$$

$$x_2F_\rho=(x_1A_{12}+x_2A_{22})V_\rho+\rho A_{23}V_z \tag{29b}$$

$$\rho F_z=(x_1A_{13}+x_2A_{23})V_\rho+\rho A_{33}V_z. \tag{29c}$$

Multiplying (29a) by x_2, (29b) by x_1 and subtracting, after some algebra we obtain

$$c_3=0 \tag{30}$$

while multiplying (29a) by x_1, (29b) by x_2 and adding, we obtain, together with (29c), the system

$$F_\rho=-(c_5+2c_4z+c_1z^2)V_\rho+\rho(c_4+c_1z)V_z \tag{31a}$$

$$F_z=\rho(c_4+c_1z)V_\rho-(c_6+c_1\rho^2)V_z. \tag{31b}$$

The integrability condition $F_{\rho z}=F_{z\rho}$ on the system (31) yields the equation

$$\rho(c_4+c_1z)(V_{\rho\rho}-V_{zz})+[(c_5-c_6)+2c_4z+c_1z^2-c_1\rho^2]V_{\rho z}-3c_1\rho V_z+3(c_4+c_1z)V_\rho=0. \tag{32}$$

At this point we suppose that $c_1\neq 0$. Then, transforming to new variables

$$\xi=c_4+c_1z \tag{33a}$$

$$\eta=c_1\rho \tag{33b}$$

and putting

$$\gamma=c_1(c_5-c_6)-c_4^2 \tag{34}$$

equation (32) becomes

$$\eta\xi(V_{\eta\eta}-V_{\xi\xi})+(\gamma+\xi^2-\eta^2)V_{\eta\xi}+3\xi V_\eta-3\eta V_\xi=0. \tag{35}$$

Equation (35) is the well known Darboux equation (Darboux 1901, Whittaker 1937). This equation, written in planar coordinates x, y, gives as a solution the total number of two-dimensional potentials $V(x,y)$ possessing an integral quadratic in the momenta, except for some special cases (Dorizzi *et al* 1983). Its solution for $\gamma\neq 0$ is

$$V=(f(u)+g(v))/(u^2-v^2) \tag{36}$$

where u and v are given by

$$2u^2 = \xi^2 + \eta^2 + \gamma + [(\xi^2 + \eta^2 + \gamma)^2 - 4\gamma\eta^2]^{1/2} \tag{37a}$$

$$2v^2 = \xi^2 + \eta^2 + \gamma - [(\xi^2 + \eta^2 + \gamma)^2 - 4\gamma\eta^2]^{1/2} \tag{37b}$$

while for $\gamma = 0$ the solution is

$$V = f(\xi^2 + \eta^2) + \frac{1}{\xi^2 + \eta^2} g(\xi/\eta) \tag{38}$$

where f and g are in both cases arbitrary functions of their respective arguments.

Equations (31) in variables ξ, η become

$$c_1 F_\eta = -(\alpha + \xi^2) V_\eta + \eta\xi V_\xi \tag{39a}$$

$$c_1 F_\xi = \eta\xi V_\eta - (\beta + \eta^2) V_\xi \tag{39b}$$

where

$$\alpha = c_1 c_5 - c_4^2 \qquad \beta = c_1 c_6$$

so that

$$\gamma = \alpha - \beta.$$

By substituting in (39)

$$c_1 F = -\beta V - G \tag{40}$$

we have

$$G_\eta = \xi(\xi V_\eta - \eta V_\xi) + \gamma V_\eta \tag{41a}$$

$$G_\xi = -\eta(\xi V_\eta - \eta V_\xi). \tag{41b}$$

The solution of equations (41) is given in Hietarinta (1986, p 30) and for $\gamma \neq 0$ it is

$$G = (v^2 f(u) + u^2 g(v))/(u^2 - v^2) \tag{42a}$$

while for $\gamma = 0$

$$G = g(\xi/\eta). \tag{42b}$$

The quadratic form in (23) becomes

$$A_{ij} p_i p_j = -(1/c_1)[2\beta T + c_1 c_2 I_1^2 + (x_1\dot{\xi} - p_1\xi)^2 + (x_2\dot{\xi} - p_2\xi)^2 + \gamma(p_1^2 + p_2^2)]$$

where $T = \frac{1}{2} p_i p_i$ is the kinetic energy, so that the second invariant I_2, after multiplying by $-c_1$ and subtracting $2\beta H + c_1 c_2 I_1^2$, becomes

$$I_2 = (x_1\dot{\xi} - p_1\xi)^2 + (x_2\dot{\xi} - p_2\xi)^2 + \gamma(p_1^2 + p_2^2) + 2\frac{v^2 f(u) + u^2 g(v)}{u^2 - v^2} \tag{43}$$

while, for $\gamma = 0$, it becomes

$$I_2 = (x_1\dot{\xi} - p_1\xi)^2 + (x_2\dot{\xi} - p_2\xi)^2 + 2g(\xi/\eta) \tag{44}$$

where ξ, η are given by (33) and u, v by (37).

For the special case $c_1 = 0$, equation (32) becomes

$$c_4\rho(V_{\rho\rho} - V_{zz}) + [(c_5 - c_6) + 2c_4 z] V_{\rho z} + 3c_4 V_\rho = 0. \tag{45}$$

We suppose that $c_4 \neq 0$ and after the transformation

$$\xi = \tfrac{1}{2}(c_5 - c_6) + c_4 z \tag{46a}$$

$$\eta = c_4 \rho \tag{46b}$$

equation (45) takes the form

$$2\xi V_{\eta\xi} + \eta(V_{\eta\eta} - V_{\xi\xi}) + 3V_\eta = 0.$$

This equation has been solved by Dorizzi *et al* (1983) for the two-dimensional case and its solution is

$$V = (1/u)[f(u+v) + g(u-v)] \tag{47}$$

where

$$u = (\xi^2 + \eta^2)^{1/2} \tag{48a}$$

$$v = \eta \tag{48b}$$

and f and g are arbitrary functions.

If we put

$$F = -c_6 V - G \tag{49}$$

equations (31) give the solution (Dorizzi *et al* 1983)

$$G = -(1/u)[(u+v)g(u-v) - (u-v)f(u+v)] \tag{50}$$

and, after subtracting suitable amounts of H and I_1^2, and dividing by two, I_2 becomes

$$I_2 = p_1(\xi p_1 - \dot{\xi} x_1) + p_2(\xi p_2 - \dot{\xi} x_2) + (1/u)[(u+v)g(u-v) - (u-v)f(u+v)] \tag{51}$$

where ξ is given by (46a) and u, v by (48).

Finally, for the degenerate case $c_1 = c_4 = 0$, equation (32) reduces to

$$V_{\rho z} = 0$$

i.e.

$$V = f(\rho) + g(z) \tag{52}$$

where f and g are arbitrary functions. System (31) yields

$$F = -c_5 f(\rho) - c_6 g(z) \tag{53}$$

and the second integral I_2 becomes

$$I_2 = c_5[p_1^2 + p_2^2 + 2f(\rho)] + c_6[p_3^2 + 2g(z)]. \tag{54}$$

At this point we may conclude the results of this section with the following.

Proposition 3. Three-dimensional integrable potentials which possess two integrals I_1, I_2, in involution, linear and quadratic in the momenta respectively, belong to one of the four classes given by equations (36), (38), (47) and (52). Integral I_1 is given by equation (27) and corresponds to the component of the angular momentum along a certain direction, while I_2 is, for each case, given by equations (43), (44), (51) and (54).

4. Conclusions

In this paper we have found the total number of three-dimensional integrable potentials which possess one integral of motion I_1 linear in the momenta and a second integral I_2 in involution with I_1 which is linear or quadratic in the momenta.

An integral of motion linear in the momenta corresponds in general to the component of the angular momentum along a certain direction and this is expressed in the potential as an ignorable coordinate in a suitably chosen system of cylindrical coordinates.

If we demand the existence of a second integral in involution, also linear in the momenta, this second integral turns out to be merely the linear momentum along the same direction. The system in fact becomes two dimensional and the potential is central.

Finally, the total number of systems which possess in addition to I_1 a second integral I_2 quadratic in the momenta is found. These potentials are found in the generic case as solutions of a Darboux equation. The two special cases treated at the end of § 3 relate also to the corresponding two-dimensional cases.

References

Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer) p 271
Bertrand M J 1852 *J. Math. Pures Appl.* **17** 121
Bountis T, Segure H and Vivaldi F 1982 *Phys. Rev.* A **25** 1257
Chadrasekhar S 1960 *Principles of Stellar Dynamics* (New York: Dover) pp 90-101
Darboux G 1901 *Arch. Neerland.* **6** 371
Dorizzi B, Grammaticos B, Hietarinta J and Ramani A 1986 *Phys. Lett.* **116A** 432
Dorizzi B, Grammaticos B and Ramani A 1983 *J. Math. Phys.* **24** 2282
Fordy A, Wojciechowski S and Marshall I 1986 *Phys. Lett.* **113A** 395
Grammaticos B, Dorizzi B, Ramani A and Hietarinta J 1985 *Phys. Lett.* **109A** 81
Hietarinta J 1986 *Direct Methods for the Search of the Second Invariant* (Turku: University of Turku)
Makarov A, Smorodinsky J, Valiev K and Winternitz P 1967 *Nuovo Cimento* A **52** 1061
Whittaker E T 1937 *Analytical Dynamics of Particles and Rigid Bodies* (Cambridge: Cambridge University Press) p 332
Yoshida H 1986 *Physica* **21D** 163
Ziglin S L 1983a *Funct. Anal. Appl.* **16** 181
—— 1983b *Funct. Anal. Appl.* **17** 6